

The Multiplicative Inverse Eigenvalue Problem over an Algebraically Closed Field

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Abstract

Let M be an $n \times n$ square matrix and let $p(\lambda)$ be a monic polynomial of degree n . Let \mathcal{Z} be a set of $n \times n$ matrices. The multiplicative inverse eigenvalue problem asks for the construction of a matrix $Z \in \mathcal{Z}$ such that the product matrix MZ has characteristic polynomial $p(\lambda)$.

In this paper we provide new necessary and sufficient conditions when \mathcal{Z} is an affine variety over an algebraically closed field.

Keywords: Eigenvalue completion, inverse eigenvalue problems, dominant morphism theorem.

1 Introduction

Inverse eigenvalue problems involving partially specified matrices have drawn the attention of many researchers. The problems are of significance both from a theoretical point of view and from an applications point of view. For background material we refer to the monograph by Gohberg, Kaashoek and van Schagen [8], the recent one by Xu [15] and the survey article by Chu [3].

The multiplicative eigenvalue problem asks for conditions which guarantee that the spectrum of a certain matrix M can be made arbitrarily through pre-multiplication by

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a matrix from a certain set. To be precise let \mathbb{F} be an arbitrary field. Let $Mat_{n \times n}$ be the space of all $n \times n$ matrices defined over the field \mathbb{F} . We will identify $Mat_{n \times n}$ with the vector space \mathbb{F}^{n^2} . Let $\mathcal{Z} \subset Mat_{n \times n}$ be an arbitrary subset and let $M \in Mat_{n \times n}$ be a fixed matrix. Then the (right) multiplicative inverse eigenvalue problem in its general form asks:

Problem 1.1. *Given a monic polynomial $p(\lambda)$ of degree n . Is there a $n \times n$ matrix $Z \in \mathcal{Z}$ such that MZ has characteristic polynomial*

$$\det(\lambda I - MZ) = p(\lambda).$$

The formulation of the left multiplicative inverse eigenvalue problem is analogous, seeking a matrix $Z \in \mathcal{Z}$ such that ZM has characteristic polynomial $p(\lambda)$. The left and the right multiplicative inverse eigenvalue problem are equivalent to each other because of the identity:

$$\det(\lambda I - ZA) = \det(\lambda I - A^t Z^t).$$

In its general form Problem 1.1 is an ‘open end problem’ and until this point only very particular situations are well understood. E.g. we would like to mention the well known result by Friedland [7] who considered the set $\mathcal{Z} = \mathcal{D}$ of diagonal matrices. Friedland did show in this case by topological methods that Problem 1.1 has an affirmative answer if the base field \mathbb{F} consists of the complex numbers \mathbb{C} . This diagonal perturbation result was later generalized by Dias da Silva [5] to situations where the base field can be any algebraically closed field.

The result which we are going to derive in this paper can be viewed as a large generalization of Friedland’s result. Specifically we will deal with the situation where $\mathcal{Z} \subset Mat_{n \times n}$ represents an arbitrary affine variety over an arbitrary algebraically closed field \mathbb{F} . Under these assumptions we will derive necessary and sufficient conditions (Theorem 3.1) which will guarantee that Problem 1.1 has a positive answer for a ‘generic set’ of matrices M and a ‘generic set’ of monic polynomials $p(\lambda)$ of degree n .

The techniques which we use in this paper have been developed by the authors in the context of the additive inverse eigenvalue problem [2, 10, 13] and in the context of the pole placement problem [12].

The major tool from algebraic geometry which we will use is the ‘Dominant Morphism Theorem’ (see Theorem 2.1). This powerful theorem necessitates that the base field is algebraically closed. The situation over a non-algebraically closed field seems to be much more complicated. Some new techniques applicable over the real numbers have been recently reported by Drew, Johnson, Olesky and van den Driessche [6].

2 Preliminaries

For the convenience of the reader we provide a summary of results which will be needed to establish the new results of this paper.

Denote by $\sigma_i(M)$ the i -th elementary symmetric function in the eigenvalues of M , i.e. $\sigma_i(M)$ denotes up to sign the i -th coefficient of the characteristic polynomial of M . Crucial for our purposes will be the *eigenvalue assignment map*

$$\psi : \mathcal{Z} \longrightarrow \mathbb{F}^n, \quad Z \longmapsto (-\sigma_1(MZ), \dots, (-1)^n \sigma_n(MZ)). \quad (2.1)$$

ψ is a morphism in the sense of algebraic geometry. By identifying a monic polynomial $\lambda^n + b_1 \lambda^{n-1} + \dots + b_n$ with the point $(b_1, \dots, b_n) \in \mathbb{F}^n$ we can also write

$$\psi(Z) = \det(\lambda I - MZ). \quad (2.2)$$

Crucial for the proof of the main result (Theorem 3.1) will be the Dominant Morphism Theorem. The following version can be immediately deduced from [1, Chapter AG, §17, Theorem 17.3].

Proposition 2.1. *Let $f : \mathcal{Z} \rightarrow \mathcal{Y}$ be a morphism of affine varieties over an algebraically closed field. Then the image of f contains a nonempty Zariski open set of \mathcal{Y} if and only if the Jacobian $df_Z : T_{\mathcal{Z}, Z} \rightarrow T_{f(Z), \mathcal{Y}}$ is onto at some smooth point Z of \mathcal{Z} , where $T_{X, \mathcal{X}}$ is the tangent space of \mathcal{X} at the point X .*

There are classical formulas, sometimes referred to as Newton formulas, which express the elementary symmetric functions $\sigma_i(M)$ uniquely as a polynomial in the power sum symmetric functions

$$p_i := \lambda_1^i + \dots + \lambda_n^i = \text{tr}(M)^i.$$

To be precise one has the formula (see e.g. [11])

$$\sigma_i(M) = \frac{1}{n!} \begin{pmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & p_1 & n-1 \\ p_n & \dots & \dots & p_2 & p_1 \end{pmatrix},$$

which induces an isomorphism $\mathbb{F}^n \rightarrow \mathbb{F}^n, (p_1, \dots, p_n) \mapsto (\sigma_1, \dots, \sigma_n)$. Based on this we equally well can study the map

$$\phi : \mathcal{Z} \longrightarrow \mathbb{F}^n, \quad M \longmapsto (\text{tr}(MZ), \dots, \text{tr}((MZ)^n)). \quad (2.3)$$

We will use the following result from [10]:

Proposition 2.2. *Let $\mathcal{L} \subset \text{Mat}_{n \times n}$ be a linear sub-space of dimension $\geq n$, $\mathcal{L} \not\subset \text{sl}_n$ (i.e. \mathcal{L} contains an element with nonzero trace). Define*

$$\pi(M) = (m_{11}, m_{22}, \dots, m_{nn})$$

the projection onto the diagonal entries. Then there exists a $S \in \text{GL}_n$ such that

$$\pi(S\mathcal{L}S^{-1}) = \mathbb{F}^n.$$

It is possible to ‘compactify’ the problem. For this consider the identity

$$\det(\lambda I - MZ) = \det \begin{bmatrix} I & Z \\ M & \lambda I \end{bmatrix}. \quad (2.4)$$

Denote by $\text{Grass}(k, n)$ the Grassmann manifold consisting of all k -dimensional linear subspaces of \mathbb{F}^n . Algebraically $\text{Grass}(k, n)$ has the structure of a smooth projective variety. In the sequel we will identify $\text{rowsp}[I \ Z]$ with a point in the Grassmannian $\text{Grass}(n, 2n)$. By identifying $\text{rowsp}[I \ Z]$ with $Z \in \text{Mat}_{n \times n}$, we can say that $\mathcal{Z} \subset \text{Grass}(n, 2n)$. Let $\bar{\mathcal{Z}}$ be the projective closure of \mathcal{Z} in $\text{Grass}(n, 2n)$. Every element in $\bar{\mathcal{Z}}$ can be simply represented by a subspace of the form $\text{rowsp}[Z_1 \ Z_2]$, where the $n \times n$ matrix Z_1 is not necessarily invertible. $\text{rowsp}[Z_1 \ Z_2]$ describes an element of \mathcal{Z} if and only if Z_1 is invertible. For any element $\text{rowsp}[Z_1 \ Z_2] \in \bar{\mathcal{Z}}$, define $\bar{\psi} : \bar{\mathcal{Z}} \longrightarrow \mathbb{P}^n$

$$\bar{\psi}([Z_1 \ Z_2]) = \det \begin{bmatrix} Z_1 & Z_2 \\ M & \lambda I \end{bmatrix}. \quad (2.5)$$

where a polynomial $b_0\lambda^n + b_1\lambda^{n-1} + \dots + b_n$ is identified with the point $(b_0, b_1, \dots, b_n) \in \mathbb{P}^n$. Recall that the Plücker coordinates of $\text{rowsp}[Z_1 \ Z_2] \in \text{Grass}(n, 2n)$ are given by the full size minors $[Z_1 \ Z_2]$, and by considering the Plücker coordinates as the homogeneous coordinates of points in \mathbb{P}^N , $N = \binom{2n}{n} - 1$, one has an embedding $\text{Grass}(n, 2n) \subset \mathbb{P}^N$ which is called Plücker embedding. Under the Plücker coordinates, (2.5) becomes

$$\bar{\psi}([Z_1 \ Z_2]) = \det \begin{bmatrix} Z_1 & Z_2 \\ M & \lambda I \end{bmatrix} = \sum_{i=0}^N z_i m_i(\lambda) \quad (2.6)$$

where $\{z_i\}$ are $n \times n$ minors of $[Z_1 \ Z_2]$ and $m_i(\lambda)$ is the cofactor of the z_i in the determinate of (2.5). $\bar{\psi}$ is undefined on the elements where

$$\det \begin{bmatrix} Z_1 & Z_2 \\ M & \lambda I \end{bmatrix} = 0.$$

So $\bar{\psi}$ is a rational map.

3 New Results

The next theorem constitutes the main result of this paper. As stated in the introduction we will identify the set $Mat_{n \times n}$ with the vector space \mathbb{F}^{n^2} and we will identify the set of monic polynomials of degree n

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n$$

with the vector space \mathbb{F}^n . If V is an arbitrary \mathbb{F} -vector space one says that $U \subset V$ forms a generic set if U contains a non-empty Zariski open subset. Over the complex or real numbers a generic set is necessarily dense with respect to the natural topology. The Dominant Morphism Theorem 2.1 states that the image of an algebraic morphism forms a generic set as soon as the linearization around a smooth point is surjective and if the field is algebraically closed.

If Problem 1.1 has a positive answer for a generic set of matrices inside $Mat_{n \times n}$ and a generic set of monic polynomials then we will say that Problem 1.1 is generically solvable. With this preliminary we have the main result of this paper:

Theorem 3.1. *Let $\mathcal{Z} \subset Mat_{n \times n}$ be an affine variety over an algebraically closed field \mathbb{F} . Then Problem 1.1 is generically solvable if and only if $\dim \mathcal{Z} \geq n$ and $\det(Z)$ is not a constant function on \mathcal{Z} .*

Proof. The conditions are obviously necessary. So we only need to prove the sufficiency. Assume that $\dim \mathcal{Z} \geq n$ and $\det(Z)$ is not a constant on \mathcal{Z} . Then there exists a curve $Z(t) \subset \mathcal{Z}$ such that

$$\frac{d}{dt} \det Z(t)|_{t=0} \neq 0,$$

$Z(0) = Z_0$ is a smooth point of \mathcal{Z} , and $\det Z_0 \neq 0$.

Let $Z(t) = Z_0 + tL + O(t^2)$ where $L \in T_{Z_0, \mathcal{Z}}$. Then

$$\det Z(t) = \det Z_0 \det(I + tZ_0^{-1}L + O(t^2)) = \det Z_0(1 + t \operatorname{tr} Z_0^{-1}L + O(t^2))$$

and

$$\frac{d}{dt} \det Z(t)|_{t=0} = \det Z_0 \operatorname{tr} Z_0^{-1}L \neq 0,$$

i.e.

$$Z_0^{-1}T_{Z_0, \mathcal{Z}} \not\subset \mathfrak{sl}_n.$$

By Proposition 2.2, there exists a $S \in Gl_n$ such that

$$\pi(SZ_0^{-1}T_{Z_0, \mathcal{Z}}S^{-1}) = \mathbb{F}^n.$$

Let

$$D := \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix} \quad (3.1)$$

and

$$M := S^{-1}DSZ_0^{-1}.$$

Then for any curve through Z_0

$$Z(t) = Z_0 + tL + O(t^2) \subset \mathcal{Z}, \quad L \in T_{Z_0, \mathcal{Z}},$$

we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\operatorname{tr}(MZ(t))^i - \operatorname{tr}(MZ_0)^i}{t} &= \lim_{t \rightarrow 0} \frac{\operatorname{tr}(MZ_0 + tML + O(t^2))^i - \operatorname{tr}(MZ_0)^i}{t} \\ &= i \cdot \operatorname{tr}((MZ_0)^{i-1}ML) \\ &= i \cdot \operatorname{tr}(D^iSZ_0^{-1}LS^{-1}). \end{aligned}$$

Let

$$V = D \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & n & \cdots & n^{n-1} \end{bmatrix} D. \quad (3.2)$$

Then V is invertible and the Jacobian $d\phi_{Z_0} : T_{Z_0, \mathcal{Z}} \mapsto \mathbb{F}^n$

$$\begin{aligned} d\phi_{Z_0}(L) &= (\operatorname{tr}(DSZ_0^{-1}LS^{-1}), 2\operatorname{tr}(D^2SZ_0^{-1}LS^{-1}), \dots, n\operatorname{tr}(D^nSZ_0^{-1}LS^{-1})) \\ &= \pi(SZ_0^{-1}LS^{-1})V \end{aligned}$$

is onto. By the Dominant Morphism Theorem 2.1, $\phi(\mathcal{Z})$ contains a nonempty Zariski open set of \mathbb{F}^n , so does $\psi(\mathcal{Z})$.

Since the set of M 's such that ψ is almost onto is a Zariski open set, and we just showed that it is nonempty, ψ is almost onto for a generic set of matrices M . \square

Next we consider the number of solutions of Problem 1.1 when $\dim \mathcal{Z} = n$. For this we introduce an important technical concept.

Definition 3.2. A matrix M is called \mathcal{Z} -nondegenerate for the right multiplicative inverse eigenvalue problem if

$$\det \begin{bmatrix} Z_1 & Z_2 \\ M & \lambda I \end{bmatrix} \neq 0 \quad (3.3)$$

for any $\operatorname{rowsp}[Z_1, Z_2] \in \tilde{\mathcal{Z}} \subset \operatorname{Grass}(n, 2n)$.

So if M is \mathcal{Z} -nondegenerate, then the map $\bar{\psi}$ defined by (2.5) becomes a morphism. In this situation we can say even quite a bit more:

Theorem 3.3. *If M is \mathcal{Z} -nondegenerate, and $\dim \mathcal{Z} = n$, then Problem 1.1 is solvable for any monic polynomial $p(\lambda)$ of degree n . Moreover, when counted with multiplicities, the number of matrices inside \mathcal{Z} which results in a characteristic polynomial $p(\lambda)$ is exactly equal to the degree of the projective variety $\bar{\mathcal{Z}} \subset \text{Grass}(n, 2n)$ when viewed under the Plücker embedding $\text{Grass}(n, 2n) \subset \mathbb{P}^N$.*

Proof. We will repeatedly use the projective dimension theorem [9, Chapter I, Theorem 7.2] which says that if X and Y are r -dimensional and s -codimensional projective varieties, respectively, then $\dim X \cap Y \geq r - s$. In particular, $X \cap Y$ is not empty if $r \geq s$.

Let

$$K = \{(z_0, \dots, z_N) \in \mathbb{P}^N \mid \sum_{i=0}^N z_i m_i(\lambda) = 0\}.$$

Then K must have co-dimension $n+1$ because of the condition $K \cap \mathcal{Z} = \emptyset$. Therefore the linear equation

$$\sum_{i=0}^N z_i m_i(\lambda) = p(\lambda). \quad (3.4)$$

has solutions in \mathbb{P}^N for any $p(\lambda) \in \mathbb{P}^n$, and the set of all solutions for each $p(\lambda)$ is in the form of $z_p + K$ where z_p is a particular solution; i.e. the solution set is given by $K_p - K$ where K_p is the unique n -codimensional projective subspace through z_p and K . Since $K \cap \bar{\mathcal{Z}} = \emptyset$, we must have

$$\dim K_p \cap \bar{\mathcal{Z}} = 0$$

and by Bézout's Theorem [14], there are $\deg \bar{\mathcal{Z}}$ many points in $K_p \cap \bar{\mathcal{Z}}$ counted with multiplicities. If $p(\lambda)$ is a monic polynomial of degree n , then from (2.5) one can see that all the solutions are in \mathcal{Z} . \square

An immediate application of Theorem 3.3 is a result of Friedland [7]: Let \mathcal{Z} be the set of all diagonal matrices. Then closure $\bar{\mathcal{Z}}$ of \mathcal{Z} inside the Grassmann variety $\text{Grass}(n, 2n)$ is isomorphic to the product of n projective lines:

$$\mathbb{P}^1 \times \dots \times \mathbb{P}^1$$

As shown in [2] the degree of $\bar{\mathcal{Z}}$ is then equal to $n!$. Moreover all points of $\bar{\mathcal{Z}}$ are of

the form $\text{rowsp}[Z_1 \ Z_2]$ where Z_1 and Z_2 are given by

$$Z_1 = \begin{bmatrix} z_{11} & 0 & \cdots & 0 \\ 0 & z_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{1n} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} z_{21} & 0 & \cdots & 0 \\ 0 & z_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{2n} \end{bmatrix}.$$

In these matrices, (z_{1i}, z_{2i}) represent the homogeneous coordinates of the i th projective line \mathbb{P}^1 .

In order to apply Theorem 3.3 we have to find the algebraic conditions which guarantee that a particular matrix M is \mathcal{Z} -nondegenerate, i.e. condition (3.3) has to be satisfied for every element $[Z_1 \ Z_2] \in \tilde{\mathcal{Z}}$. For this let I be a subset of $\{1, 2, \dots, n\}$, J be the complement of I , and $|J|$ be the number of elements in J . For any point $[Z_1 \ Z_2] \in \tilde{\mathcal{Z}}$, assume

$$\begin{aligned} z_{1i} &= 0 & \text{for } i \in I, \\ z_{1j} &\neq 0 & \text{for } j \in J. \end{aligned}$$

Without loss of generality we can take

$$\begin{aligned} z_{2i} &= 1 & \text{for } i \in I, \\ z_{1j} &= 1 & \text{for } j \in J, \end{aligned}$$

and (2.5) becomes

$$\bar{\psi}([Z_1 \ Z_2]) = \pm M_I \lambda^{|J|} + \text{lower power terms},$$

where M_I is the principal minor of M consisting of the i th rows and columns, $i \in I$. Furthermore if we take

$$z_{2j} = 0 \text{ for } j \in J$$

then

$$\bar{\psi}([Z_1 \ Z_2]) = \pm M_I \lambda^{|J|}.$$

Therefore M is \mathcal{Z} -nondegenerate if and only if all the principal minors of M are nonzero. So we have Friedland's result [7, Theorem 2.3] formulated for an algebraically closed field: If all the principal minors of M are nonzero, then the multiplicative inverse eigenvalue problem with perturbation from the set of diagonal matrices is solvable for any monic polynomial $p(\lambda)$ of degree n , and there are $n!$ solutions, when counted with multiplicities.

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